ARTICLE NO. 79-1051R

Two-Phase Mach Number Description for Equilibrium Duct Flow of Nitrogen

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For equilibrium two-phase flow the squared ratio of mixture volume to mixture sound speed, $\beta(g,T)$ is shown to have the same form as many weighted mean two-phase properties; namely $\beta(g,T)=g\beta(1,T)+(1-g)\beta(0,T)$, where g is the liquid mass fraction and $\beta(1,T)$ and $\beta(0,T)$ are the isothermal saturated liquid and vapor values of β which are generated for nitrogen in tabulated form by a computer program. With these β tables a simplified method of calculating two-phase Mach numbers is developed for various duct flows. One- and two-phase Mach number jumps at phase boundaries are also discussed.

Nomenclature

A	= duct cross-sectional area
$a,\bar{a}$	= one- and two-phase sound speed, respectively
c_p	= one-phase specific heat at constant pressure
$c_v,\bar{c}_v$	= one- and two-phase specific heat at constant volume, respectively
e	= specific value of a generalized thermodynamic function defined by Eq. (6)
G	= mass flux = V/v
g	= liquid mass fraction
h	= specific enthalpy
$k,\bar{k}$	= one- and two-phase isentropic expansion coefficients, respectively
M	= one-phase Mach number = V/a
$\bar{M}$	= two-phase Mach number = $V/\bar{a}$
P	= pressure
s	= specific entropy
T	= temperature
V	= flow velocity
v	= specific volume
x	= rectangular coordinate
β	= squared ratio of specific volume to sound speed
γ	= ratio of one-phase specific heats = c_p/c_v
φ	= specific Gibbs function

Superscripts

$(\ )'$	= $\mathrm{d}/\mathrm{d}T$
$(\ )''$	= $\mathrm{d}^2/\mathrm{d}T^2$
$(\ )^*$	= value of a property at Mach 1 state

Subscripts

G	= saturated vapor
L	= saturated liquid

Introduction

THE cryogenic wind tunnel program at the Langley Research Center uses nitrogen as the test gas and achieves a significant increase in the Reynolds number as the temperature of the test gas is decreased. In addition, the increase in Reynolds number is not associated with any increase in dynamic pressure and is obtained with a reduced actual fan power.[1-4] Ultimately, the Reynolds number increase is limited by partial liquefaction of the test gas, and the onset of condensation transforms the original single-phase flow problem to the more difficult one involving the flow of two phases. Mathematical models for two-phase flow come in varying degrees of complexity[5-8] with the simplest assuming homogeneous equilibrium. This model is most useful in a modest pressure gradient situation associated with a time scale conducive to achieving local equilibrium between the phases. In contrast to this, onset of condensation may be delayed in a high gradient flow only to be followed by a rapid irreversible condensation in a short distance of travel.[5] In addition, other types of nonequilibrium may occur.[6]

Even though flow with homogeneous phase equilibrium represents a simplified model, application difficulties can lead to the use of approximate models containing additional simplifications. On the one hand, a solution can be obtained which combines thermodynamic property tables for both one- and two-phase regions of the test fluid with appropriate flow equations in a computer program as in Ref. 7. This approach, while cumbersome, nevertheless yields exact information on the interplay of the flow variables. On the other hand, an elegant example of an approximate technique which bypasses computer use of entire sets of thermodynamic property tables is illustrated in a paper by Wegener and Mack.[5] In their procedure, various approximations are used in order to generate closed-form solutions minimally dependent on thermodynamic property tables. The approximations restrict the range of validity of the solution to the triple point region.

This paper retains the generality of Collins'[7] approach. However, it does not combine the thermodynamic property tables and the flow equations into one large computer program related to one particular kind of duct flow as the isentropic nozzle flow, for example. Instead, it first identifies additional thermodynamic properties of the usual two-phase form linear in the moisture fraction, which are found to be useful in the analysis of many kinds of duct flow in addition to isentropic nozzle flow; for example, Rayleigh flow, Fanno flow, and shocks. The major role of the computer is to use the nitrogen thermodynamic property formulation of Jacobsen[9] to generate auxiliary tables of the saturated liquid-vapor values for these additional functions. With these, the various

Presented as Paper 79-1051 at the AIAA 14th Thermophysics Conference, Orlando, Fla., June 4-6, 1979; submitted Dec. 26, 1979; revision received April 14, 1980.

Index categories: Multiphase Flows; Nozzle and Channel Flow.

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duct flow problems can then be addressed separately from a different and, it is hoped, simpler base amenable to solution with hand calculators. The method depends on knowing two saturation properties as functions of temperature, namely, vapor pressure and Gibbs function. With these, additional two-phase functions linear in moisture fraction are generated which ultimately reveal that the defined function $\beta=(v/\bar{a})^2$ can be written as

$$\beta(g,T)=g\beta_L(T)+(1-g)\beta_G(T) \tag{1}$$

where v is the mixture specific volume, $\bar{a}$ the equilibrium two-phase sound speed, g the moisture fraction, and the two temperature functions on the right side are the saturated liquid and vapor values of β, respectively, namely, $\beta(1,T)$ and $\beta(0,T)$. Both are known and tabulated.

For use with steady duct flow, the two-phase Mach number

$$\bar{M}=V/\bar{a} \tag{2}$$

where V is the flow velocity, and the mass flux

$$G=V/v \tag{3}$$

are used with the definition of β to give the two-phase Mach number for duct flows as

$$\bar{M}^2=G^2\beta(g,T) \tag{4}$$

For all of these flows, the right side of Eq. (4) will consist of constants of the respective flows and combinations of properties of the form of Eq. (1) whose saturated liquid and vapor portions are tabulated as functions of temperature. The moisture fraction is also determined as an implicit function of temperature by the respective thermodynamic paths used to describe the respective flows. This makes $\bar{M}^2$ in Eq. (4)a known function of temperature which can be computed readily by use of the tables. All of the previously indicated duct flows can have an associated choking phenomenon which is now simply described by setting the local two-phase Mach number equal to one. Denoting this type of choked state by an asterisk, Eq. (4) then gives

$$G^{*2}\beta^*=1 \tag{5}$$

The thermodynamic paths for any of these duct flows convert this to a complicated equation in the single unknown T^* which has to be solved by iteration. With the extended thermodynamic properties, including the saturated values of β, the iteration is done relatively easily using a small hand calculator.

Two-Phase Properties

In the subsequent treatment of various two-phase flows, weighted mean values of certain thermodynamic properties will be needed. They have the common form

$$e(g,T)=ge_L(T)+(1-g)e_G(T) \tag{6}$$

where $e_L(T)$ and $e_G(T)$ are the saturated liquid and vapor values $e(1,T)$ and $e(0,T)$. If the value of the mixture property e is known at any state, then the moisture fraction can be calculated by solving Eq. (6) to give

$$g=(e_G-e)/(e_G-e_L) \tag{7}$$

provided the temperature is also known for that state and the tables of saturation values of $e_L(T)$ and $e_G(T)$ are available. This, of course, is the common case for e representing any of the specific quantities—entropy, enthalpy, and volume.

As already indicated, it will subsequently be shown that the new β function can also be included in the list of functions having the generalized form of e. This will be possible only because the two-phase specific heat at constant volume also has this generalized form, as will now be shown.

The exact differential of the Gibbs function is

$$d\varphi=v dP-s dT \tag{8}$$

Since φ and P are functions of temperature, this can be written as

$$s(T,v)=vP'(T)-\varphi'(T) \tag{9}$$

From this it follows that

$$\bar{c}_v(T,v)=T(vP''-\varphi'') \tag{10}$$

This is transformed to a function of moisture fraction and temperature by substituting Eq. (6) for v. This gives

$$\bar{c}_v(g,T)=T\{gv_L(T)+(1-g)v_G(T)\}P''(T)-T\varphi''(T) \tag{11}$$

On any isotherm the saturation boundary values of $\bar{c}_v$ occur at $g=0$ and $g=1$. These are

$$\bar{c}_{v,G}(T)=\bar{c}_v(0,T)=T(v_GP''-\varphi'') \tag{12}$$

$$\bar{c}_{v,L}(T)=\bar{c}_v(1,T)=T(v_LP''-\varphi'') \tag{13}$$

When the first of these is multiplied by $(1-g)$ and the second by g, their sum is equal to the right side of Eq. (11); therefore,

$$\bar{c}_v(g,T)=g\bar{c}_{v,L}(T)+(1-g)\bar{c}_{v,G}(T) \tag{14}$$

which is the same linear form as the generalized Eq. (6). Now it remains to evaluate the right-hand sides of Eqs. (12) and (13).

At the Langley Research Center, the computer program for nitrogen properties utilizes thermodynamic data from Jacobsen.[9] In his formulation, saturation boundary values for specific volume, entropy, and enthalpy are given, as is the equation for the vapor pressure as a function of temperature from which P' and P'' are obtained. The two-phase Gibbs function is obtained by substitution of appropriate h and s boundary values into the defining equation for this function, namely

$$\varphi=h-Ts \tag{15}$$

A computer program is then devised to give the first and second derivatives $\varphi'(T)$ and $\varphi''(T)$. Finally, the boundary values of the two-phase specific heat at constant volume are computed from Eqs. (12) and (13). This method of computing the boundary values from the two-phase side is not unique, of course. Another method will be outlined in the next section when various jump conditions at the saturation boundary will be discussed.

The thermodynamic two-phase sound speed is computed from

$$\bar{a}^2=-v^2(\partial P/\partial v)_s \tag{16}$$

assuming equilibrium between the phases; i.e., it is the so-called zero frequency speed of sound.[5] In order to facilitate its use with the Langley computer programming of nitrogen properties, it is transformed as follows. After forming ds from Eq. (9), Eq. (10) is used in the result to give

$$ds(T,v)=\frac{\bar{c}_v}{T}dT+P'dv \tag{17}$$

Table 1 Heat capacity and sonic velocity jumps

T, K	$\bar{c}_{v,L}$	$c_{v,L}$	$\bar{c}_{v,G}$	$c_{v,G}$	$\bar{a}_L$	a_L	$\bar{a}_G$	a_G
80	2029.2	981.33	18251	776.43	3.75	894	163	177
100	2178.5	934.92	10205	841.75	16.80	613	167	183
120	2868.5	958.84	6314	987.40	49.00	338	137	176

which is then transformed to $ds(P,v)$ by replacing dT by $dP/P'(T)$; that is

$$ds(P,v) = \frac{\bar{c}_v}{TP'}dP + P'dv \tag{18}$$

This gives

$$\left(\frac{\partial P}{\partial v}\right)_s = -\frac{T(P')^2}{\bar{c}_v} \tag{19}$$

which, when substituted into Eq. (16) transforms the square of the two-phase equilibrium sound speed into

$$\bar{a}^2 = T(vP')^2/\bar{c}_v \tag{20}$$

a form now compatible with prior programming parameters.

For comparison, the one-phase counterpart of Eq. (20) is

$$a^2 = -\gamma v^2\left(\frac{\partial P}{\partial v}\right)_T \tag{21}$$

where γ is the one-phase heat capacity ratio c_p/c_v. Substitution for c_p from the thermodynamic equation for the difference in specific heat capacities gives the comparable single phase form of Eq. (20) as

$$a^2 = \frac{Tv^2}{c_v}\left(\frac{\partial P}{\partial T}\right)_v^2 - v^2\left(\frac{\partial P}{\partial v}\right)_T \tag{22}$$

A jump in c_v also occurs and this can be used to write a different computer program to evaluate $\bar{c}_{v,G}$ and $\bar{c}_{v,L}$ from that previously discussed. The jump relation can be derived by setting ds from Eq. (17) equal to ds from its one-phase counterpart along the saturation boundary to obtain

$$\bar{c}_{v,G} - c_{v,G} = -T\left(\frac{\partial P}{\partial v}\right)_{T,G}(v_G')^2 \tag{23}$$

where v_G' is dv_G/dT. A corresponding jump is obtained for saturated liquid by changing the subscript from G to L.

Some heat capacity and sonic velocity jumps are shown in Table 1 for nitrogen, with c_v, $\bar{c}_v$ in J/kg-K and a, $\bar{a}$ in m/s.

Two-Phase β Function

When Eq. (20) is used in the definition of β,

$$\beta = (v/a)^2 \tag{24}$$

and $\bar{c}_v$ is replaced by Eq. (14), the result is an expression for β linear in g:

$$\beta(g,T) = \frac{g\bar{c}_{v,L}(T)}{T(P')^2} + \frac{(1-g)\bar{c}_{v,G}(T)}{T(P')^2} \tag{25}$$

from which the isothermal boundary values are evaluated as

$$\beta_L(T) = \beta(1,T) = \frac{\bar{c}_{v,L}(T)}{T\{P'(T)\}^2} \tag{26}$$

Table 2 Saturation value of β[a]

T, K	$\beta_L \times 10^8$	$\beta_G \times 10^8$	T, K	$\beta_L \times 10^8$	$\beta_G \times 10^8$
87	3.6623	26.127	98	0.92740	0.6146
88	3.1791	21.928	99	0.83162	4.0143
89	2.7707	18.476	100	0.74272	3.5010
90	2.4240	15.627	101	0.67306	3.0604
91	2.1279	13.266	102	0.60745	2.6819
92	1.8742	11.300	103	0.54942	2.3551
93	1.6560	9.6592	104	0.49813	2.0727
94	1.4672	8.2829	105	0.45266	1.8278
95	1.3033	7.1248	106	0.41241	1.6148
96	1.1609	6.1471	106.2	0.40491	1.5757
97	1.0364	5.3188	107	0.37671	1.4295

[a] β units in $[(m^2\text{-}s)/kg]^2$.

which is also $\beta_L(T) = (v_L/\bar{a}_L)^2$, and

$$\beta_G(T) = \beta(0,T) = \frac{\bar{c}_{v,G}(T)}{T\{P'(T)\}^2} \tag{27}$$

which is also $\beta_G(T) = (v_G/\bar{a}_G)^2$. Thus, β can be written in the compact form of Eq. (1), which now joins the list of thermodynamic functions contained in the general form of Eq. (6). Its boundary values for nitrogen are shown in Table 2.

Isentropic, Two-Phase, Nozzle Flow

For isentropic, homogeneous, two-phase flow in a horizontal converging-diverging duct, the differential cross-sectional area change dA is related to the velocity differential dV by the same form of equation used to describe single-phase flow. This form will be of importance later in defining conditions at the saturated vapor interface of a nozzle expansion that originates in the single-phase region and terminates in the two-phase region. The conditions will also be related to the discontinuity in the sound speeds at the phase boundary, hence to the corresponding discontinuity in the one- and two-phase Mach numbers which arises because of the assumption of a continuous flow velocity at the interface.

The single- and two-phase forms are, respectively,

$$\frac{dA}{A} = (M^2 - 1)\frac{dV}{V} \tag{28}$$

$$\frac{dA}{A} = (\bar{M}^2 - 1)\frac{dV}{V} \tag{29}$$

It is seen from these that for accelerating flow in a converging duct, the flow is subsonic; in a diverging duct, the flow is supersonic. For decelerating flow in a converging duct, the flow is supersonic; in a diverging duct, the flow is subsonic. The throat condition is ordinarily described by a Mach number of one. An exception to this is described by Collins,[7] who shows that the theoretical choking of single component, isentropic, homogeneous equilibrium flows from highly subcooled liquid states do not choke at the sonic condition of $M=1$ or $\bar{M}=1$. The conditions that produce this effect at the saturated liquid boundary of water are found at the saturated vapor boundary also and will be illustrated for combined single- and two-phase expansions of nitrogen.

Suppose that the saturated vapor state occurs at 100 K and the flow velocity at this state is 175 m/s. From Table 1, $M_G = 0.956$ and $\bar{M}_G = 1.048$. With no change in velocity, the flow has changed from subsonic to supersonic in passing through the saturated vapor state; but now it is referenced to the two-phase Mach number. This is analogous to the situation described by Collins[7] for water taken from a highly subcooled state to a low-pressure saturated liquid state where discontinuous choking occurs, in that $(dG/dP)_L$ is discontinuous at that state. This derivative is written for one-phase use as

$$\frac{dG}{dP} = \frac{M^2 - 1}{V} \tag{30}$$

Similarly, for the two-phase region,

$$\frac{d\bar{G}}{dP} = \frac{\bar{M}^2 - 1}{V} \tag{31}$$

When the expanding flow reaches the saturated vapor curve denoted by the subscript G (not to be confused with the mass flux), the jump in dG/dP is

$$\left(\frac{d\bar{G}}{dP}\right)_G - \left(\frac{dG}{dP}\right)_G = \left(\frac{\bar{M}^2 - M^2}{V}\right)_G \tag{32}$$

which is equivalent to Collins[7] equation (A-4), except that his equation is written for the saturated liquid jump. Substituting the numerical values already given for V_G, $\bar{M}_G$ and M_G into Eq. (32) gives the jump at the saturated vapor point of $+0.001054$ for this contrived example.

To illustrate further, an example follows for isentropic flow of nitrogen in a nozzle which uses the tabulated β function in conjunction with the usual tables for specific volume, entropy, and enthalpy. In this example, calculations began after the discontinuous choke at $g = 0$, with an emerging supersonic two-phase Mach number and an assumed gradient $dP/dx < 0$ in the remainder of the diverging section. Here, x is the coordinate along the axis of the duct in the flow direction. Reservoir conditions, denoted by an 0 subscript, are arbitrarily selected at a temperature of 125 K and a pressure of 2×10^6 N/m^2. At this state, thermodynamic property tables[9] give $v_0 = 0.014021$ m^3/kg, $h_0 = 101489$ J/kg, and $s_0 = 4887.8$ J/kg-K. The isentrope from the reservoir state intersects the saturated vapor curve in a state designated by the subscript 1 for which $v_{G1} = 0.020477$ m^3/kg, $h_{G1} = 87644.84$ J/kg, $\bar{a}_{G1} = 163.812$ m/s, $a_{G1} = 182.866$ m/s, and $T_1 = 106.27$ K. Applying the energy equation between the two states gives the flow velocity as $V_1 = 166.398$ m/s. From this and the two sound velocities, a pair of Mach numbers is obtained as $\bar{M}_{G1} = 1.0158$ and $M_{G1} = 0.9099$.

As previously mentioned, the two-phase flow calculations begin from the two-phase side of the saturated vapor state which exists in the nozzle throat, and are continued by successively decreasing choices of temperature. For each choice of temperature, usual two-phase computations involving various functions selected from Eq. (6) are made. Moisture fraction is computed from the isentrope as $(s_G(T) - 4887.8) \div (s_G(T) - s_L(T))$ and this is used in Eq. (6) to calculate $v(g,T)$, $h(g,T)$, and $\beta(g,T)$, using Table 2 for the necessary saturation values of β and Jacobsen's property formulation[9] for the others. The flow velocity is obtained from

$$V = (2(h_0 - h_{G1}))^{1/2} \tag{33}$$

Using this and v in Eq. (3) gives the mass flux which is substituted together with β into Eq. (4) in order to obtain $\bar{M}$. Finally, the continuity equation is used to give the area ratio

$$A/A_{G1} = vV_{G1}/v_{G1}V \tag{34}$$

The results shown in Table 3 were obtained by using an ordinary hand calculator to program the necessary data from the previously discussed tables of saturation values for v, h, s, and β.

Using the saturated vapor section Mach numbers $M_{G1} = 0.9099$ and $\bar{M}_{G1} = 1.0158$ in Eqs. (28) and (29) will give two $(dA/dx)_{G1}$ values of opposite sign if a nonzero velocity gradient is assumed. Since this cannot be for a smooth nozzle, both $(dA/dx)_{G1}$ and $(dV/dx)_{G1}$ must be zero. This is reflected in the Table 3 computation. From Euler's equation, the throat pressure gradient $(dP/dx)_{G1}$ is also zero. The same conclusions are reached for a saturated liquid state $(g = 1)$ occurring in the throat. In this latter case, partial experimental verification is suggested for initially subcooled nitrogen in runs 1333 and 1334 in Fig. 11a and Fig. 10 of Refs. 10 and 11, although interpretation difficulties arise from the use of a short, constant area section to represent the throat.

For a doubly subsonic saturated liquid or saturated vapor state in the converging part of the nozzle or for a doubly supersonic state in the diverging section, Eqs. (28), (29), and Euler's equation require discontinuous velocity and pressure gradients for smooth ducts. This effect is not apparent in the experimental data of the last two references. In some instances, this may be a manifestation of metastable effects. In Ref. 10, it is pointed out that a nonequilibrium model[12] is more reliable for stagnation temperatures in the liquid region that are below the critical temperature. The homogeneous model is preferred for stagnation temperatures at or above the critical. For stagnation temperatures much greater than critical and stagnation entropies greater than critical, only the homogeneous model is applicable. For nitrogen wind tunnel stagnation states, metastable effects may again appear. When a flow expands in a nozzle from a superheated reservoir to a pressure below saturation, it encounters a surface energy barrier associated with droplet formation[5] as the flow reaches the saturated vapor state. This barrier acts to delay the onset of condensation, unless the flow contains some sort of impurity droplets or particles to serve as condensation sites for the vapor. Smaller flow gradients enhance the chance of droplet formation upon available condensation sites and, to this degree, aid in achieving homogeneous equilibrium. Whether or not a particular flow expansion approximates homogeneous equilibrium, the model can give the researcher

Table 3 Choked nozzle with a subsonic-supersonic throat

T, K	$g \times 10^2$	h, J/kg	v, m^3/kg	V, m/s	$[\beta \times 10^8]$[a]	$\bar{M}$	A/A_G	P/P_0
106.269	0.00000	87644.8	0.020477	166.398	1.56256	1.0158	1.00000	0.5868
106.200	0.08034	87544.2	0.020547	167.002	1.57475	1.0196	1.00017	0.5844
106.000	0.31956	87238.6	0.020782	168.822	1.61095	1.0310	1.00035	0.5772
105.000	1.44149	85747.2	0.021977	177.436	1.80793	1.0856	1.00649	0.5422
104.000	2.50430	84230.1	0.023583	185.790	2.03331	1.1391	1.01728	0.5087
102.000	4.45185	81159.0	0.026122	201.643	2.58952	1.2422	1.05270	0.4463

[a] β units in [m^2-s/kg]^2.

an idea of the magnitudes of effects to be expected; consequently, an idea of the appropriate type of detection instrumentation needed. Goglia and Van Wylen[13] found 1.94—to 6.83 K degrees of supersaturation for condensing temperatures of nitrogen from 70 to 60.56 K. Again, as with liquid, the supersaturation effect is strongest at the lower temperature, dropping as the critical point with its zero surface tension/low surface energy barrier is approached.

As previously indicated, the homogeneous model requires $(dP/dx)_{GI}$ to be discontinuous for a saturated vapor state occurring in the diverging part of the nozzle. A similar discontinuity may exist in the pressure gradient at a point downstream of the throat, which defines onset of condensation after metastable flow. This can be seen in Fig. 5 of Ref. 13.

When Eq. (6) is used for v and β and these are incorporated into the defining equation for the isentropic expansion coefficient, numerical calculations then show that this coefficient can be approximated as a constant for short expansions of nitrogen. From this it is possible to obtain an approximate formula for critical pressure which greatly expedites the usual throat computations.

Combining the definition of the two-phase isentropic expansion coefficient

$$\bar{k} = -\frac{v}{P}\left(\frac{\partial P}{\partial v}\right)_s \tag{35}$$

with Eqs. (16) and (24) results in

$$\bar{k} = v/P\beta \tag{36}$$

With v and β both linear in g and both possessing tabulated saturation values, $\bar{k}$ is readily investigated on an isentrope $s=c$ by first calculating $g=(s_G-c)/(s_G-s_L)$ at any pressure. This is then substituted into the appropriate form of Eq. (6) for v and β before inserting these into Eq. (36) for $\bar{k}$.

Starting with a saturated vapor state at 92.63 K whose entropy is 5108.3 J/kg-K, it was found that $\bar{k}$ varied on the isentrope from its saturated value of 1.171 to 1.16 at 72 K. A similar order of magnitude variation in k was observed for other isentropic expansions starting at various points on the saturated vapor boundary and extending into the coexistence region for several degrees. Therefore, this suggests treating $\bar{k}$ as constant for short, isentropic expansions of nitrogen in the two-phase region. Under these conditions, Eq. (35) integrates into

$$Pv^{\bar{k}} = c = P_1 v_1^{\bar{k}} \tag{37}$$

where state 1 is an arbitrary two-phase reference state. When $(\partial P/\partial v)_s$ is eliminated between Eqs. (16) and (35), the square of the sound speed is

$$\bar{a}^2 = \bar{k}Pv \tag{38}$$

and this is used with Eq. (37) to write the sound speed in terms of the reference state as

$$\bar{a}^2 = \bar{a}_1^2 \bar{r}^{(\bar{k}-1)/\bar{k}} \tag{39}$$

where $\bar{r}=P/P_1$. Euler's equation is integrated by first eliminating v by Eq. (37). Use of Eq. (38) then gives

$$\bar{r} = \left\{\left[1+\frac{\bar{k}-1}{2}\bar{M}_1^2\right]\bigg/\left[1+\frac{\bar{k}-1}{2}\bar{M}^2\right]\right\}^{\bar{k}/(\bar{k}-1)} \tag{40}$$

As with other substances, it is also found for nitrogen that the single plane isentropic expansion coefficient in the superheat region varies slowly enough for useful representation as an approximate constant. With this assumption, taking the reference state on the isentrope as the saturated vapor point, the array of Eqs. (37-40) will represent the single-phase expansion when the bars over the symbols are dropped throughout. Then the single-phase version of Eq. (40) becomes

$$r = \left\{\left[1+\frac{k-1}{2}M_1^2\right]\bigg/\left[1+\frac{k-1}{2}M^2\right]\right\}^{k/(k-1)} \tag{41}$$

Here M_1 and $\bar{M}_1$ are the fixed Mach number pair at the saturated vapor state on the isentrope and M and $\bar{M}$ are the respective one- and two-phase variable Mach numbers; r is P/P_1. If the variable state in Eq. (41) is taken as the reservoir state, then $r=r_0$, where

$$r_0 = P_0/P_1 \tag{42}$$

For the case where the saturated vapor state exists in the diverging portion of the nozzle, a continuous one-phase choke state is established in the throat. Denoting this state by an asterisk and setting $M=1$ in Eq. (41) gives

$$r^* = \frac{P^*}{P_1} = \left[\frac{2}{k+1}\left(1+\frac{k-1}{2}M_1^2\right)\right]^{k/(k-1)} \tag{43}$$

For the overall pressure ratio from reservoir to reference plane, M is set equal to zero in Eq. (41) to give, after using Eq. (42),

$$r_0 = \frac{P_0}{P_1} = \left(1+\frac{k-1}{2}M_1^2\right)^{k/(k-1)} \tag{44}$$

The critical pressure is then obtained from r^*/r_0 as

$$\frac{P^*}{P_0} = \left(\frac{2}{k+1}\right)^{k/(k-1)} \tag{45}$$

When the location of the saturated vapor point is taken to its limiting position at the throat, P_1 becomes equal to P^* and Eq. (43) requires M_1 to be equal to one. This would represent the limit of continuous one-phase choking.

If the saturated vapor state exists in the converging portion of the nozzle, a continuous two-phase choke state will exist in the throat. This time setting $\bar{M}=1$ in Eq. (40) gives

$$\bar{r}^* = \frac{P^*}{P_1} = \left[\frac{2}{\bar{k}+1}\left(1+\frac{\bar{k}-1}{2}\bar{M}_1^2\right)\right]^{\bar{k}/(\bar{k}-1)} \tag{46}$$

Now the critical pressure is given by $\bar{r}^*/r_0$ as

$$\frac{P^*}{P_0} = \left[\frac{2}{\bar{k}+1}\left(1+\frac{\bar{k}-1}{2}\bar{M}_1^2\right)\right]^{\bar{k}/(\bar{k}-1)}\bigg/\left[1+\frac{k-1}{2}M_1^2\right]^{k/(k-1)} \tag{47}$$

Again, when the limiting position of the saturated vapor point is moved into the throat, $P^*=P_1$, and Eq. (46) gives $\bar{M}_1=1$. This gives the limit of two-phase continuous choking.

Horizontal Pipe Flow

Since G is constant for pipe flow, the two-phase Mach number can be determined from β using Eq. (4). Once g is known from an appropriate path relation, Eq. (1) substituted in Eq. (4) determines the two-phase Mach number as a function of temperature. As with the nozzle, introduction of both one- and two-phase Mach numbers creates problems in joining the respective solutions at phase boundaries. To show this, a general derivation independent of the number of phases present will be made relating the differential entropy and pressure changes for Rayleigh and Fanno flows. The bar notation above certain two-phase symbols will not be used in the derivation until it again becomes necessary to distinguish one-phase relations from two-phase relations.

Combining the differential of $P(s,v)$ with the one-phase form of Eq. (16) and rearranging gives

$$ds=\left(\frac{\partial s}{\partial P}\right)_v\left(dP+\frac{a^2}{v^2}dv\right) \tag{48}$$

Then the Euler equation is combined with the continuity equation with $dA=0$. This gives

$$dv=-(v^2/V^2)dP \tag{49}$$

Substituting this and the Mach number definition into the expressions for the entropy differential gives

$$ds=\left(\frac{\partial s}{\partial P}\right)_v\left(\frac{M^2-1}{M^2}\right)dP \tag{50}$$

From single-phase thermodynamics

$$\left(\frac{\partial s}{\partial P}\right)_v=\frac{c_v}{T}\left(\frac{\partial T}{\partial P}\right)_v \tag{51}$$

and the corresponding two-phase form is obtained from Eq. (18) as

$$\left(\frac{\partial s}{\partial P}\right)_v=\frac{\bar{c}_v}{TP'} \tag{52}$$

The phase question is made explicit now by reverting to the bar notation and using the last two equations in Eq. (50) to give

$$ds=\frac{c_v}{T}\left(\frac{\partial T}{\partial P}\right)_v\frac{M^2-1}{M^2}dP \tag{53}$$

for single-phase use, and the following for the two-phase region

$$ds=\frac{\bar{c}_v}{TP'}\frac{\bar{M}^2-1}{\bar{M}^2}dP \tag{54}$$

Along the phase boundary, c_v, $\bar{c}_v$, $(\partial T/\partial P)_v$ and P' are all positive and both equations, when referred to their respective Mach numbers, require characteristic Rayleigh flow behavior. For heat addition, ds must be positive; therefore, subsonic flow is accompanied by pressure drop and acceleration, and supersonic flow by pressure rise and deceleration. When the flow regime is completely one phase or completely two phase, choking will occur at $M=1$ or $\bar{M}=1$. When the flow is part single phase and part two phase, with a saturated state possessing two different Mach numbers separating the two flow regimes, joining problems similar to that of the nozzle may be present. As a theoretical example, if a flow velocity can be generated with a value lying between the two sound velocities at the saturated state separating the two regimes, this would not only preclude any Mach 1 state there, but would also cause the $(M-1)$ and $(\bar{M}-1)$ as well as the dP terms in Eqs. (53) and (54) to be of opposite sign.

The calculation routine for obtaining a Mach number representation of a two-phase pipe flow begins by integrating Eq. (49) for a constant, known G to give $P+G^2v=c$. This gives v as a function of P (or T). At this temperature, g is determined from Eq. (7) with v taken as e, substituted into Eq. (1) to give β. Finally, β and G are used in Eq. (4) to give $\bar{M}$.

In adiabatic Fanno flow, the path relationship that yields g as a function of temperature (or pressure) is more complicated. For known stagnation enthalpy and mass flux, the energy and continuity equations are combined to give $(h+G^2v^2/2)=h_0$. The enthalpy and specific volume are replaced by using Eq. (6) with e alternately taken as h and v. This leads to a quadratic equation for g whose solution is

$$g=\frac{h_{LG}+G^2v_Gv_{LG}}{G^2v_{LG}^2}\left[1-\sqrt{1-\frac{G^2v_{LG}^2\{G^2v_G^2-2(h_0-h_G)\}}{(h_{LG}+G^2v_Gv_{LG})^2}}\right] \tag{55}$$

Here $h_{LG}(T)$ is used for $h_G(T)-h_L(T)$ and $v_{LG}(T)$ is likewise used for $v_G(T)-v_L(T)$. Since all of the variables on the right side of this equation are tabulated functions of temperature, g is also known as a function of temperature on the two-phase Fanno line. With G and β known, the two-phase Mach number is then calculated from Eq. (4). For a two-phase choked flow, the above iteration procedure is used until the choking condition $\bar{M}=1$ is reached.

When the flow regime is split between a two-phase portion on one side of a saturation boundary plane, say $g=0$, and a superheat portion on the other side, discontinuities in the Mach number again appear. To illustrate the effect of these jump conditions, a differential equation relating entropy and pressure will be obtained for Fanno flow.

Combining the continuity and energy equations gives

$$dh=-G^2vdv \tag{56}$$

and this is used to eliminate dh is the combined first and second law expression

$$Tds=dh-vdp \tag{57}$$

to give

$$Tds=-G^2vdv-vdp \tag{58}$$

Using $G=V/v$, eliminating ds between Eqs. (48) and (57), and then introducing the Mach number definition, gives

$$dv=-\frac{v^2}{a^2}\left\{\left[T\left(\frac{\partial s}{\partial P}\right)_v+v\right]\Big/\left[T\left(\frac{\partial s}{\partial P}\right)_v+M^2v\right]\right\}dP \tag{59}$$

Substitution of this for dv in Eq. (48) results in the generalized expression

$$ds=v\left\{(M^2-1)\Big/\left[T+v\left(\frac{\partial P}{\partial s}\right)_vM^2\right]\right\}dP \tag{60}$$

This is now specialized for use with one-phase flow by substituting Eq. (51) for $(\partial P/\partial s)$. This gives

$$Tds=v\left\{(M^2-1)\Big/\left[1+\frac{v}{c_v}\left(\frac{\partial P}{\partial T}\right)_vM^2\right]\right\}dP \tag{61}$$

The two-phase form is obtained by substituting Eq. (52) into Eq. (60) to give

$$Tds=v\left\{(\bar{M}^2-1)\Big/\left[1+\frac{vP'}{\bar{c}_v}\bar{M}^2\right]\right\}dP \tag{62}$$

Some implications of Eqs. (61) and (62) are illustrated with respect to the following example where a supersonic, two-phase Fanno flow is diffused to $\bar{M}=1$ which occurs at the $g=0$ section. For flow parameters $G=2529.68$ kg/m^2-s and $h_0=99143.6$ J/kg, the results are shown above the row of asterisks in Table 4. This shows that pressure and entropy increase as supersonic Mach number decreases, in conformity with Eq. (62) whose right-hand sign is determined by the product of $(\bar{M}^2-1)$ and dP, since all of the other terms are positive.

Ordinary single-phase choking in Fanno flow is associated with a Mach 1 section at the end of the pipe. In mixed one- and two-phase Fanno flow, Eqs. (61) and (62) seem to be able to accommodate two such Mach 1 planes. In the example, the one-phase Mach number associated with the $g=0$ plane is calculated as 0.921. If the flow emerging from this plane is still pipe flow, it is now one-phase, subsonic pipe flow. In order to satisfy Eq. (61), it would require a negative dP since the right-hand sign is determined by the product $(M^2-1)\,dP$. From Eqs. (59), (51), and the continuity equation, a negative dP accelerates the flow until the choking limit of $M=1$ is reached at the end of the pipe. Thus the assumption of one-phase flow after the $g=0$ section can result in choking with $\bar{M}=1$ and $M=1$ both required, as far as Eqs. (61) and (62) are concerned.

If the flow parameters G and h_0 could be adjusted to make the original supersonic two-phase flow enter the $g=0$ section still slightly supersonic and leave subsonic, no Mach 1 state would be associated with the $g=0$ section and the end of the duct would have the one-phase $M=1$ section in choked flow. Again, in order to have this flow, the pressure would first increase in the direction of flow and then decrease. Then the original example could be considered a limiting case of this type of flow.

A possible alternative is to have the incoming supersonic, two-phase flow reach a $g=0$ section at the end of the pipe with $\bar{M}=1$ from the inner side and $M<1$ just outside the pipe end.

Since the $g=0$ section is characterized by $M=\bar{a}\bar{M}/a$, another possibility is to have both $\bar{M}>1$ and $M>1$ there. For such a section located away from the pipe end, Eqs. (61) and (62) require a pressure rise throughout the pipe in the direction of flow. With two-phase flow upstream of this section, any downstream choking would be associated with the single-phase $M=1$.

The entries below the row of asterisks in Table 4 represent a subsonic Fanno flow entering the two-phase region at a $g=0$ section for flow parameters $G=2794.7$ kg/s-m^2 and $h_0=94359$ J/kg. The flow velocity is 120 m/s at this section and the one-phase sound speed is 183 m/s. These give $M=0.6557$ at the $g=0$ section where the table shows a corresponding $\bar{M}=0.71352$ at the same section. Hence a subsonic one-phase flow occurs upstream of the $g=0$ section and a subsonic two-phase flow occurs downstream of this section. Equations (61) and (62) require a falling pressure in the direction of flow in both regions.

The variety of problems in Fanno flow associated with the Mach number discontinuity at a $g=0$ section also appear in Rayleigh pipe flow.

Such a flow is illustrated in Table 5. The flow parameters are set at a 90 K, $g=0$ section with $V=\bar{a}_G=167.7$ m/s and $G=2529.68$ kg/s-m^2. This produces a Mach number discontinuity at the section with $\bar{M}=1$ and $M=0.9214$.

For heat input ds must be positive and Eq. (54) requires a rising pressure in the direction of flow for the two-phase supersonic domain. This is confirmed in the section of the table above the row of asterisks, reading from top down. The same Rayleigh line has a two-phase subsonic branch which is tabulated below the row of asterisks in Table 5. Now pressure falls in the direction of flow, again satisfying Eq. (54), until a second $g=0$ state is reached which has the pair of Mach numbers $\bar{M}=0.8928$ and $M=0.8240$. Thus, both one-phase Mach numbers associated with the two $g=0$ states are subsonic and Eq. (53) indicates a one-phase, subsonic Rayleigh branch connecting them, going from the $g=0$ state at 91.39 K to the one at 90 K, with M increasing from 0.8240 to $M=0.9214$. Continuation on this branch will terminate the subsonic portion at $M=1$. Then, if the supersonic two-phase flow above the asterisk row is joined to this one-phase subsonic segment, a double Mach 1 situation is again potentially possible at the cost of a theoretical rising/falling pressure pattern in the flow direction.

Discussion

Since the emphasis in this paper has been on calculating two-phase Mach numbers for these various flows using the newly defined β function, no effort has been made to compute the associated one-phase flows which are to be joined to corresponding single-phase flows at $g=0$ sections. Because of this arbitrary limit on the scope of the paper, only tentatively held theoretical speculations about the behavior of the associated one-phase flows at $g=0$ sections in Rayleigh and Fanno flows have been offered in order to indicate a range of possible solutions associated with the Mach number discontinuities at these sections.

Even though this paper stressed Mach number discontinuities at the $g=0$ section of duct flow, the basic equations are applicable to the $g=1$ saturated liquid section as well, and similar problems in joining one- and two-phase flows will be encountered there.

In deciding to feature the new β function in these two-phase Mach number computations, it has already been pointed out that the major role of the computer is to calculate its saturated liquid and vapor values. Once these are known, β can then be applied directly to all of the previously discussed duct flows, without further reference to the auxiliary functions (Gibbs function and the vapor pressure equation) necessary for its construction. In order to re-emphasize the usefulness of these functions, the vapor pressure equation and the Gibbs function will be used for calculating the specific volume as a function of temperature in two-phase Fanno flow. To do this, entropy is eliminated between Eqs. (9) and (10) to give

$$h=\varphi-T\varphi'+TvP' \tag{63}$$

and this is substituted into the energy equation $h_0=h+G^2v^2/2$ for the enthalpy. This gives a quadratic in v whose solution is

$$v=\frac{TP'}{G^2}\left[-1+\sqrt{1-\frac{2G^2(\varphi-T\varphi'-h_0)}{(TP')^2}}\right] \tag{64}$$

For known flow parameters G and h_0, the right side is a known function of temperature, since $P'(T)$, $\varphi(T)$, and

Table 4 Fanno flows

T, K	P, N/m^2	g	$\beta\times10^8$, m^{4}s^2/kg^2	$\bar{M}$, m/s	s, J/kg K
87	276500	0.033463	23.375	1.2743	5136.8
88	302770	0.020617	21.541	1.1741	5146.6
89	330850	0.009531	18.326	1.0829	5151.8
90	360790	0.000000	15.627	1.0000	5153.5
*	*	*	*	*	*
95.6	566400	0.00000	6.5186	0.71352	5058.6
95	541070	0.002688	7.1092	0.7451	5063.6
94	500700	0.007690	8.2305	0.8176	5070.9
93	462550	0.013471	9.5514	0.86370	5076.5

Table 5 Rayleigh flow

T, K	P, N/m^2	g	$\beta\times10^8$, m^{4}s^2/kg^2	$\bar{M}$	s, J/kg K
87	276500	0.06908	24.575	1.2540	5061.1
88	302770	0.03785	21.218	1.1653	5110.7
89	330850	0.01422	18.253	1.0808	5142.3
90	360790	0.00000	15.627	1.0000	5154.3
*	*	*	*	*	*
94	500700	0.08466	7.706	0.7022	4929.3
93	462550	0.03678	9.365	0.7741	5032.7
92	426570	0.00871	11.218	0.8473	5102.1
91.39	405645	0.0000	12.843	0.8928	5129.2

$\varphi'(T)$ were evaluated by the computer program in the course of computing the saturated values of β. From this equation, v is computed and used in Eq. (7) as e to compute g, and this quantity was calculated previously by using Eq. (55) simply because its right side is expressed in the readily available and familiar tabulation of v_L, v_G, h_L, and h_G. For basically the same reason, β was chosen as a convenient variable to use in calculating two-phase Mach numbers. First, it fits into the well-known two-phase form given by Eq. (6). Second, it absorbs the less familiar two-phase functions P', P'', φ, φ', and φ''. This may make computations easier with the β function, but at the same time, it may obscure the overall importance of the Gibbs function and the vapor pressure curve equation in the description of duct flow, including high-temperature heat pipes.[14]

Acknowledgment

The paper was written as a partial result of the activities of the principal author while a summer research fellow at the Langley Research Center. The fellowship program was under the joint auspices of NASA and ASEE acting through the administrative auspices of the Old Dominion University.

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